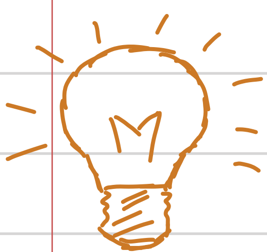
Thick Subcategory Theorem.

<u>Ultimate Goal</u> State & prove the thick subcat theorem (TST).

<u>Outline</u> I. Motivation



II. Start with TST

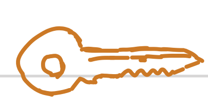
- Def of cats $\mathcal{C}\Gamma$ & FH (consistent with notion in [Rav 92])
  Def of thick subcat.
- Statement of TST

1. algebraic TST
2. geometric (classical) TST.

II. 1. algebraic TST

- Landweber Filtration Thm
- Generalization to $\mathcal{P}(n)$, abelian cat of $P(n)_* P(n)$-comodule finitely presented as $P(n)_*$-mod
  - ▶ Review of various spectra related to BP.
- Proof of TST, algebraic version.

II. 2. classical TST.

- Basics in Sp, hpy direct limit / homotopy colim.
- Nilpotence thm, smash product form, and how it is derived from the classical nilpotence thm.
- Statement of <u>key corollary</u> 
- Properties of Spanier-Whitehead duality, intuition about S-W duality & additional notes.
- Back to TST. Intro & prove <u>Lem</u> ★
- Before the proof of the key corollary, we need the background in Bousfield classes.

Basics about thm. Proof of class invariance thm under the assumption of TST. Structure of <BP>, including Johnson-Yosimura thm.

- Proof of the key corollary & TST.

Tools to help us (All taken for granted!)

- Landweber Filtration Thm
- Thm by Morava - Landweber on invariant prime ideals of BP_*
- Theory of Bousfield classes.
- Nilpotence Thm.

Background

- Spectra
- Bousfield classes
- S-W duality.

Reference

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I. Motivation.

<u>Recall</u> $X \in Sp$. p-local finite, type n. $K(n)_*(X) \neq 0$. $n = \infty$.

$K(n)$ Morava K. satisfies.

- Künneth formula $K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_* Y$.
- $K(n)_*(X) = 0 \Rightarrow K(n-1)_*(X) = 0$.
- $K(n)_* X = K(n)_* \otimes H_*(X; \mathbb{Z}_{(p)})$.
- $K(n)_* = \mathbb{Z}/p [v_n, v_n^{-1}]$.

$\mathbb{Z}_{(p)}$ p-adic integers.

$A_p = \bigvee_{n \geq 0} K(n)$, $A = \bigvee_{p \text{ prime}} A_p$. $X \in Sp$. is said to be harmonic, if
(p prime)
it is A_*-local. It is dissonant if it A_*-acyclic. Then $(A_s)_* X = \mathbb{Q}$, $s \neq p$. Only $(A_p)_*$-locality and acyclicity.

[Rav 84] BP harmonic.

If X connective spectrum, finite type, hom $\dim_{MU_*} MU_* X$ (minimal length of a resolution of $MU_* X$ by proj. graded MU_*-mod), X is harmonic. In particular, X finite $\Rightarrow$ X harmonic.

- Non-trivial finite p-local spectrum X has a type.

e.g. $\mathbb{S}$ type 0.

$\mathbb{S}/p$ type 1.

$\mathrm{cofib}(\mathbb{S}/p \to \mathbb{S}/p)$ type 2. denoted $\mathbb{S}/(p, v_1)$.

$v_1 : \Sigma^{2(p-1)} \mathbb{S}/p \longrightarrow \mathbb{S}/p$. p odd.

induces $K(1)_*$-iso. i.e. $K(1)_*(v_1)$ iso, multiplication by v_1.

$v_2 : \Sigma^{2(p^2-1)} \mathbb{S}/(p,v_1) \longrightarrow \mathbb{S}/(p,v_1)$ $p \geq 5$.

This leads to

<u>Periodicity Conj.</u> p-local finite spectrum X type n. $\exists$ v_n-map $v_n : \Sigma^? \backslash \longrightarrow X$ inducing $K(n)_*$-iso. given by the multiplication by some power of v_n.

$\Rightarrow$ Can construct a lots of elets in $\pi_* \mathbb{S}$.

<u>Realizability Conj.</u> $\forall\, I \subset BP_*$ I "looks like" invariant prime ideal $I_n = (p, v_1, \cdots, v_n)$, $\Rightarrow$ $\mathbb{S}/I_n$ admits a v_n-map.

e.g. $p=2$. $v_2^{32} : \Sigma^{192} \mathbb{S}/(2,v_1) \longrightarrow \mathbb{S}/(2,v_1)$

$K(2)_*$-homlgy iso, $\cdot v_2^{32}$ [BHHM 08]

$p=3$. $v_2^9 : \Sigma^{144} \mathbb{S}/(3,v_1) \longrightarrow \mathbb{S}/(3,v_1)$

$K(2)_*$-homlgy iso, $\cdot v_2^9$ [BP 04]

Not clear $\mathbb{S}/(p.v_1)$ admits a v_2-map. not nilpotent. We need to find some spectrum detecting nilpotence. i.e. R ring spectrum, Hurewicz map $\pi_* R \xrightarrow{h} E_* R$, $\ker h$ consists nilpotent elets.

<u>Nilpotence Conj.</u> MU does that!

Goro Nishida thm: $\forall \alpha \in \pi_k \mathbb{S}$, $k>0$, α nilpotent. Sketch: $MU_* \mathbb{S} = MU_* = \mathbb{Z}[b_1, b_2, \dots]$. $|b_i| = 2i$, torsion-free. $\pi_* \mathbb{S}$ torsion in every >0 degree, $h=0$, when $k>0$. So $\forall$ positive deg elet α is nilpotent.

If $W \to X \to Y$ cofib seq, $f: Y \to \Sigma W$. $MU_*(f^{(k)}) = 0$, then $\langle X \rangle = \langle W \rangle \vee \langle Y \rangle$. Bousfield class, $E \in Sp$.

$$\langle E \rangle = \{ X \in Sp : E \wedge X = 0 \}$$

Class Invariance Conj X, Y p-local finite, X type m. Y type n.
then $\langle X \rangle = \langle Y \rangle \Leftrightarrow m = n$.

Leads to telescope conjecture. $f: X \to \Sigma^{-k} X$, $X \in Sp$ type n.
$f^{-1} X = \operatorname{colim} (X \xrightarrow{f} \Sigma^{-k} X \xrightarrow{f} \Sigma^{-2k} X \xrightarrow{f} \cdots)$

Telescope Conj $\langle f^{-1} X \rangle$ depends only on n.
$\langle f^{-1} X \rangle = \langle K(n) \rangle$ or wedge of $\langle K(n) \rangle$.

e.g. (Johnson - Yosimura) $\langle v_n^{-1} BP \rangle = \langle E(n) \rangle = \bigvee \langle K(n) \rangle$.

helps to prove $L_n X \simeq X \wedge L_n \mathbb{S}$. L_n = localization w.r.t. $E(n)$.
(Johnson - Wilson Thy) $E(n) = v_n^{-1} BP / (v_{n+1}, v_{n+2}, \dots)$

Devinatz - Smith - Hopkins proves all but telescope conj. Telescope Conj is still open.

$$\text{Nilpotence} \Longrightarrow \text{TST} \Longrightarrow \begin{cases} \text{Class invariance} \\ \text{Periodicity} \Longrightarrow \text{Realizability.} \end{cases}$$

Our Job is to prove TST.

II. Notations

<u>Def.</u> Γ = gp of power series / $\mathbb{Z}$ $\gamma \in \Gamma$.

$$\gamma = x + b_1 x^2 + b_2 x^3 + \cdots$$
$$= \sum_{i=1}^{\infty} b_{i-1} x^i \qquad b_0 = 1.$$

gp operation = multiplication.

L = Lazard ring $L \cong MU_* \cong \mathbb{Z}[b_1, b_2, \cdots]$, $|b_i| = 2i$. $\exists$ universal f.g.l. over L. of the form $G(x,y) = \sum_{i,j} a_{ij} x^i y^j$. $a_{ij} \in L$.

s.t. $\forall$ f.g.l. F/R. R comm. unital ring $\exists!$ ring homphism $\theta : L \to R$

s.t. $F(x,y) = \sum_{i,j} \theta(a_{ij}) x^i y^j$.

Let $\gamma \in \Gamma$. $\gamma^{-1}(G(\gamma(x), \gamma(y)))$ is another f.g.l. $/L$

$\Rightarrow \begin{cases} \phi : L \to L \\ \gamma \text{ invertible in } \Gamma \end{cases} \Rightarrow \phi : L \to L$ automorphism.

$\Rightarrow$ Γ-action on L.

- Steenrod alg. $\sum_{n \geq 0} Sq^n$ mod 2.

<u>Def.</u> $C\Gamma$ = cat. of. finitely presented, graded L-mods M.

+ Γ-action compatible with its action on L

$$\underset{L\text{-mod.}}{M} \circlearrowleft^{\Gamma} \circlearrowright L$$

FH = cat of finite spectra, $[-,-]_*$.

$\overline{MU_*} : FH \longrightarrow C\Gamma$.

<u>Recall</u> F f.g.l. $/R$. $n \in \mathbb{Z}$.

n-series $[n](x) = F(x, [n-1](x)) = \underbrace{x +_F x +_F \cdots +_F x}_{n}$

$[1](x) = x$

$[-n](x) = i([n](x))$.

• F multiplicative, $[n](x) = (1+x)^n - 1$

ht of F at p. $\quad ht_p F = h$, $\quad [p](x) \equiv a x^{p^h} + \cdots \mod p$.

$ht_p F = \infty$, $\quad [p](x) \equiv 0 \mod p$.

Write $I_{p,n} \subset L$, $I_{p,n} = (p, v_1, v_2, \ldots, v_{n-1})$. $\quad I_{p,0} = 0$

$I_{p,\infty} = (p, v_1, \ldots)$.

If fix p, then denoted as I_n.

Thm (Morava-Landweber).

The only Γ-invariant prime ideals in L are $I_{p,n}$. Moreover, $n>0$, the subgp of $L/I_{p,n}$ fixed by Γ is $\mathbb{Z}/p[v_n]$.

Thm (Landweber Filtration Theorem).

$\forall$ mod in $\mathcal{CT}$, M, M admits a finite filtration by submods in $\mathcal{CT}$,

$$0 = M_s \subset \cdots \subset M_2 \subset M_1 \subset M_0 = M$$

s.t. $\forall\ 0 \le i \le s-1$, $M_i / M_{i+1} \cong MU_* / I_{n_i}$, $\quad I_{n_i} = (p, v_1, \ldots, v_{n_i - 1})$.

stable iso. i.e. after a dimension shift.

If localize at p, only matters are v_n. $\mathcal{CT}$ Γ-action on $L = MU_*$.

No analogue on BP_*! One need to replace Γ by some groupoid; Hopf alg $\rightsquigarrow$ Hopf algebroids.

Def. $\mathcal{C}$ full subcat of $\mathcal{CT}$. It is called thick, if it satisfies

(algebraic version) $\quad 0 \to M' \to M \to M'' \to 0$ s.e.s.

then $M \in \mathcal{C} \iff M', M'' \in \mathcal{C}$.

Def. $\mathcal{D}$ full subcat of FH. It is called thick, if it satisfies.

1) cofiber seq. $X \xrightarrow{f} Y \to C_f$

(geometric version). $\quad$ 2-out-of-3.

2) $X \vee Y \in \mathcal{D} \Rightarrow X, Y \in \mathcal{D}$.

• Th. (Algebraic TST).

Let $\mathcal{C}$ thick subcategory of $\mathcal{CT}_{(p)}$ (abelian cat of all BP_*BP-comodule finitely presented as BP_*-mod). Let $\mathcal{C}_k$ = full subcat of $\mathcal{CT}_{(p)}$ s.t. $v_{k-1}^{-1}M = 0$, $\forall M \in \mathcal{C}_k$. Then $\mathcal{C} = \mathcal{C}_k$, for some $k \geq 0$.

• Th (Geometric TST)

$\mathcal{D}$ thick subcat of $FH_{(p)}$ (cat of p-local finite spectra). Then $\mathcal{D} = \mathcal{D}_k$, where $\mathcal{D}_k$ = full subcat of $FH_{(p)}$ s.t. $v_{k-1}^{-1}\overline{MU}_*(X) = 0$. for some $k \geq 0$.

• $FH_{(p)} = \mathcal{D}_{p,0} \supset \mathcal{D}_{p,1} \supset \mathcal{D}_{p,2} \supset \cdots$

$\mathcal{CT}_{(p)} = \mathcal{C}_{p,0} \supset \mathcal{C}_{p,1} \supset \mathcal{C}_{p,2} \supset \cdots$

inclusions are strict. [Mit 85] by Mitchell.

MU_* sends $\mathcal{D}_{p,n} \to \mathcal{C}_{p,n}$.

III. Proof of algebraic TST.

Recall BP.

Johnson - Wilson spectra $BP\langle n\rangle$. It is obtained from BP by killing $(v_{n+1}, v_{n+2}, \cdots) \subset BP_*$.

$$\Rightarrow \quad \pi_* BP\langle n\rangle = \mathbb{Z}_{(p)}[v_1, \ldots, v_n].$$

v_n - map : $\Sigma^{2(p^n-1)} BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \longrightarrow BP\langle n-1\rangle$.

• $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$.

v_n : $BP\langle n\rangle \xrightarrow{v_n} \Sigma^{-2(p^n-1)} BP\langle n\rangle \xrightarrow{v_n} \cdots$

$E(n) = \operatorname{colim} \Sigma^{-2i(p^n-1)} BP\langle n\rangle$.

$= v_n^{-1} BP\langle n\rangle$.

$E(0) = H\mathbb{Q}$. $\quad E(n)_* = \mathbb{Z}_{(p)}[v_1, \cdots, v_{n-1}, v_n^{\pm}]$.

Morava K - theory $K(n) = \operatorname{colim} \Sigma^{-2i(p^n-1)} k(n)$.

where connective spectrum $k(n)$. by killing $(p, v_1, \cdots, v_{n-1}, v_{n+1}, v_{n+2}, \cdots)$.

$$\pi_* k(n) = \mathbb{Z}_{(p)}[v_n].$$

fibration $\Sigma^{2(p^n-1)} k(n) \xrightarrow{v_n} k(n) \longrightarrow H\mathbb{Z}_{(p)}$.

$$k(0) = H\mathbb{Z}_{(p)} = BP\langle 0 \rangle.$$

$P(n)$ obtained by BP by killing $(p, v_1, v_2, \ldots, v_{n-1}) = I_n$

So $P(0) = BP$ $\quad \pi_* P(n) = BP_*/I_n$.

$$B(n) = \mathrm{colim}\,(\text{———}).$$

<u>Thm</u> (Morava - Landweber)

- I_n invariant. Only invariant prime ideals in BP. $0 \le n \le \infty$.
- s.e.s. $0 \to \Sigma^{2(p^n-1)} \underbrace{BP_*/I_n}_{P(n)_*} \xrightarrow{v_n} \underbrace{BP_*/I_n}_{P(n)_*} \longrightarrow \underbrace{BP_*/I_{n+1}}_{P(n+1)_*} \to 0$

E_* = comm. $P(n)_*$-alg s.t. $E_* \otimes_{P(n)_*} (-)$ is exact in $\mathcal{P}(n)$

abelian cat of $P(n)_* P(n)$-comodule f.p. as $P(n)_*$-mod.

$n = 0$. $\mathcal{P}(n) = \mathcal{CP}_{(p)}$.

[Rnd 86] $E_* \otimes_{P(n)_*} P(n)_*(-)$ homology thy. e.g. of LEFT.

$E_*E = E_* \otimes_{P(n)_*} P(n)_* P(n) \otimes_{P(n)_*} E_*$. Can make it (E_*, E_*E) a Hopf algebroid by extending the structure maps. in $P(n)_* P(n)$. E_*E is flat $/ E_*$. b/c $P(n)_* P(n)$ flat $P(n)_*$-mod. $\forall N$ E_*-mod.

$$E_*E \otimes_{E_*} N = E_* \otimes_{P(n)_*} P(n)_* P(n) \otimes_{P(n)_*} N.$$

Let $M \in \mathcal{P}(n)$. $E_* \otimes_{P(n)_*} M$ is an E_*E-comodule b/c

$$M \longrightarrow P(n)_* P(n) \otimes_{P(n)_*} M \longrightarrow E_*E \otimes_{P(n)_*} M$$
$$\longrightarrow E_*E \otimes_{E_*} (E_* \otimes_{P(n)_*} M)$$

$\mathcal{J}$ = cat of obj. $E_* \otimes_{P(n)_*} M$. $M \in \mathcal{P}(n)$ (Fix n).

mor. $E_* \otimes f$. $f : M_1 \to M_2 \in \mathcal{P}(n)$

- $E_* \otimes_{P(n)_*} (-)$ exact on $\mathcal{S}$.
  $P(n) \longrightarrow \mathcal{S}$.

$\mathcal{S}_k = (E_* \otimes_{P(n)_*} (-))(P(n)_k)$.

$\Rightarrow \quad \cdots \mathcal{S}_{k+1} \subseteq \mathcal{S}_k \subset \cdots \subset \mathcal{S}_n = \mathcal{S}$.

not. nec. strict.

<u>Generalized TST</u> If $\mathcal{C}$ thick subcat of $\mathcal{S}$, then $\mathcal{C} = \mathcal{S}_k$.

Proof of TST

<u>Recall</u> Landweber Filtration tells us

$\forall M \in \mathcal{CT}_{(p)}$. admits finite filtration

$$0 = M_s \subset \cdots \subset M_1 \subset M_0 = M$$

s.t. $M_i / M_{i+1} \underset{\text{stable}}{\cong} BP_* / I_{n_i} \qquad 0 \le i \le s-1$.

Fix p. $\forall M \in \mathcal{CT}_{(p)}$. $\operatorname{Spec} M = \{0\} \cup \{m \ge 1 : v_{m-1}^{-1} M = 0\}$, $v_0 = p$.

If $M \ne 0$, then $\operatorname{Spec} M$ = finite subset of $\mathbb{N}$.

⚠ $v_{m-1}^{-1} M = 0 \iff K(m-1)_* M = 0$. (Talk later !)

Let $\mathcal{C}$ = thick subcat of $\mathcal{CT}_{(p)}$. $\bigcap_{M \in \mathcal{C}} \operatorname{Spec} M = \{0, 1, 2, \ldots, k\}$

$k = \max \bigcap_{M \in \mathcal{C}} \operatorname{Spec} M$.

$\mathcal{C} \subset \mathcal{C}_k$. $\mathcal{C} \not\subset \mathcal{C}_{k+1}$ ($K(n)_*(X) = 0 \Rightarrow K(n-1)_*(X) = 0$).

$M \in \mathcal{C}$ s.t. $v_{k-1}^{-1} M = 0$ but $v_k^{-1} M \ne 0$.

Let $0 = M_s \subset \cdots \subset M_1 \subset M_0 = M$ be the finite filtration by LFT. By def of $\mathcal{C}$, $M_i \in \mathcal{C}$. $M_i / M_{i+1} \in \mathcal{C}$.

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_i / M_{i+1} \longrightarrow 0$$

localization $\Rightarrow$ exact functor, preserving

$\Rightarrow \quad n_i \ge k, \quad \forall\, 0 \le i \le s-1 \qquad (*)$.

$BP_* / I_{n_i} = \mathbb{Z}_{(p)}[v_{n_i}, v_{n_i+1}, \ldots]$.

On the other hand, by $v_k^{-1} M \ne 0$. $\Rightarrow \exists j$. s.t. $v_k^{-1} \dfrac{BP_*}{I_{n_j}} \ne 0$.

$\exists\ n_j \le k, \quad 0 \le j \le s-1 \quad (**)$.

$BP_*/I_{n_j} \underset{\text{stable}}{\simeq} M_i/M_{i+1}$, $M \in \mathcal{C}$. $M_i/M_{i+1} \in \mathcal{C}$. $M_i \in \mathcal{C}$.

$\forall\ n_i \ge k, \quad 0 \le i \le s-1.$

$\Rightarrow \exists\ n_\ell, \quad \ell \in [0, s-1]$

$n_\ell = k$

$\Rightarrow BP_*/I_{n_\ell} = BP_*/I_k \in \mathcal{C}$.

Consider. e.s. in $\mathcal{CT}_{(p)}$: $\quad r \ge 0$

$$0 \to BP_*/I_{k+r} \xrightarrow{\cdot v_{k+r}} BP_*/I_{k+r} \longrightarrow BP_*/I_{k+r+1} \to 0$$

$r = 0$, $\quad BP_*/I_{k+1} \in \mathcal{C}$.

$r > 0$, induction $\Rightarrow BP_*/I_{k+r} \in \mathcal{C}. \quad r \ge 0$.

Rk. At this pt, we actually prove that $\mathcal{C}_k \subset \mathcal{C}_{k+1}$. strict.

$\mathcal{C} \subset \mathcal{C}_k$. Suffice to prove $\mathcal{C}_k \subset \mathcal{C}$:

$\forall N \in \mathcal{C}_k$. Landweber Filtration

$$0 = N_s \subset \cdots \subset N_1 \subset N_0 = N.$$

we know $v_{k-1}^{-1} N = 0 \Rightarrow n_i \ge k \quad \forall\ 0 \le i \le s-1$

$N_i/N_{i+1} \underset{\text{stable}}{\simeq} BP_*/I_{n_i}$

$N_s = 0 \in \mathcal{C}$. By s.e.s.

$$0 \to \underset{\underset{\mathcal{C}.}{\cap}}{\underline{N_{i+1}}} \to N_i \longrightarrow N_i/N_{i+1} = \underset{\underset{\mathcal{C}.}{\cap}}{BP_*/I_{n_i}} \to 0.$$

By induction $\Rightarrow N_i \in \mathcal{C}$.

$\Rightarrow N_0 = N \in \mathcal{C}. \Rightarrow \mathcal{C}_k \subset \mathcal{C}$.

$\Rightarrow \mathcal{C} = \mathcal{C}_k$.

IV. Classical TST. / Geometric TST.

1. Review of Sp.

ring spectrum E unit map : $\eta: \mathbb{S} \to E$

multiplication map : $m: E \wedge E \to E$.

s.t. ------

module spectrum M/E $\mu: E \wedge M \to M$.

E ring spectrum. Call E is flat if $E \wedge E \simeq \bigvee \Sigma^? E$

X_i, $i \geq 1$. Sp. coproduct $\bigvee_{i>0} X_i$. $\pi_* \bigvee X_i = \bigoplus \pi_* X_i$

$$E \wedge (\bigvee X_i) = \bigvee (E \wedge X_i).$$

$$E_* (\bigvee X_i) = \bigoplus_{i>0} E_* X_i$$

$\forall X \in Sp$. $f: F \to X$, F finite. Regard this as an obj. in a new cat $\mathcal{G} = \{(F_i, f_i)\}$ associated with X.

$(F_1, f_1) \longrightarrow (F_2, f_2)$ s.t.

$$\begin{tikzcd} F_1 \arrow[rr, "g \in Sp."] \arrow[dr, "f_1"'] & & F_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

Any pair $f_i: F_i \to X$, $i = 1, 2$. can factor through $f: F_1 \vee F_2 \longrightarrow X$

$\exists$ canonical map $\operatorname{colim} F_\alpha \xrightarrow{\alpha} X$.

<u>Prop</u>. $\forall X$, α is w.e.

<u>Cor</u>. $\forall$ finite (CW) spectrum, is htpy colimit of its finite subspectra.

2. Nilpotence theorem.

<u>Classical Nilpotence</u> R is a connective ring spectrum, $\pi_* R \xrightarrow{h} MU_* R$

Then $\alpha \in \pi_* R$ is nilpotent if $h\alpha = 0$.

<u>Nilpotence Thm, smash product form</u>

$f: F \to X$, $F, X \in Sp$. F finite. f is smash nilpotent if $MU \wedge f$ is null-homotopic.

(Localize at p, $BP \wedge f$)

▶ Classical $\Rightarrow$ Smash product version:

$f: F \to X$, F finite. adjoint to

$$\hat{f}: \mathbb{S} \longrightarrow X \wedge DF$$

Aside. $Hom(V, W) \cong V^* \otimes W$.

$E \wedge f$ null htpic (E ring spectrum). $\Longleftrightarrow$ $E \wedge \hat{f}$ is

It suffices to prove $F = \mathbb{S}$, nilpotence thm, smash product version.

$F = \mathbb{S}$. $f: \mathbb{S} \to X$. Claim f is smash nilpotent if

$$\begin{tikzcd} \mathbb{S} \arrow[r, "f"] & X \arrow[r, "\eta\wedge id"] \arrow[d, equal] & E \wedge X \\ & \mathbb{S} \wedge X & \end{tikzcd}$$

is null-htpic.

Since X = htpy colim of its finite subspectra X_α. both f, composite factor thrgh X_α, X_α finite, i.e.

$$\mathbb{S} \xrightarrow{f} X_\alpha \xrightarrow{\eta\wedge id} E \wedge X_\alpha \quad \text{is nilpotent.}$$

$Y = \Sigma^n X_\alpha$ w/ n s.t. Y is 0-connected spectrum. $R = \bigvee_{j \geq 0} Y^{(j)}$

$\Rightarrow$ classical nilpotence tells us $\alpha \in \pi_* R$ nilpotent if $h\alpha = 0$

$h: \pi_* R \longrightarrow MU_* R$.

$E = MU$. Then

$$\begin{tikzcd} \mathbb{S} \arrow[r, "f"] & X_\alpha \arrow[r, "\eta\wedge id"] & MU \wedge X_\alpha \arrow[d, squiggly, "\Sigma^n -"] \\ \Sigma^n \mathbb{S} \arrow[r, "f"] & \Sigma^n X_\alpha \arrow[r, "\eta\wedge id"] \arrow[d, equal] & MU \wedge Y. \\ & Y & \end{tikzcd}$$

$\Big\downarrow$ according def of R.

$\pi_* R \longrightarrow \pi_*(MU \wedge R) = MU_* R$ null-htpic.

$\Rightarrow$ f corresponds to some nilpotent elet $\alpha \in \pi_* R$

s.t. $h\alpha = 0$.

$\Rightarrow$ f is smash nilpotent.

Key Corollary

Let W, X, Y be p-local finite spectra, $f: X \to Y$, then $W \wedge f^{(k)}$ null-homotopic for $k \gg 0$, if $K(n)_*(W \wedge f) = 0$, $\forall n \geq 0$

3. Spanier-Whitehead duality. (SW duality)

X finite spectrum. The following properties are what we need:

- **Thm.** $\exists !$ finite spectrum DX (S-W duality of X) s.t.
  1) $D^2X \simeq X$. $[X, Y]_* \simeq [DY, DX]_*$.
  2) E_* homology theory, $E_*X \cong E^{-*}DX$.
  3) Y finite spectrum, $D(X \wedge Y) \simeq DX \wedge DY$.
  4) $X \longmapsto DX$, $D(-)$ contravariant.
  5) $\forall Y \in Sp$, $[X, Y]_* \cong \pi_*(DX \wedge Y)$

     In particular, $D\mathbb{S} \simeq \mathbb{S}$.

     - $S^n \longrightarrow DX \wedge Y$, $\Sigma^n X \to Y$ adjoint.

       $X = Y$, $id \dashv \underline{e} : S \longrightarrow DX \wedge X$
  6). $Hom_{K(n)_*}(K(n)_*(X), K(n)_*(Y)) \cong K(n)_*(DX \wedge Y)$.

     $\forall n$. ($K(n)_*(X)$ free over $K(n)_*$).

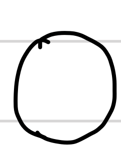
- Geometric intuition.

$X \subset S^n$ cpt. locally contractible. $X \neq S^n, \phi$.

Alexander duality: $\tilde{H}^{n-i-1}(X; \mathbb{Z}) \cong \tilde{H}_i(S^n \setminus X; \mathbb{Z})$.

<u>Fatal Drawback</u>: X does not det. $S^n \setminus X$.

$X = S^1$, $n = 3$ $K = S^n \setminus X$ is a knot.



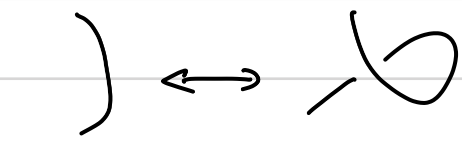
unknotted

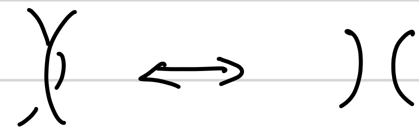
0

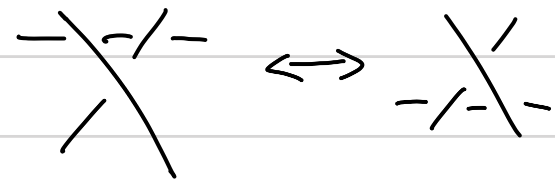
right-handed trefoil 3_1

left-handed trefoil 3_2.

$3_1 \not\simeq 3_2$ by Reidemeister moves.

R1 

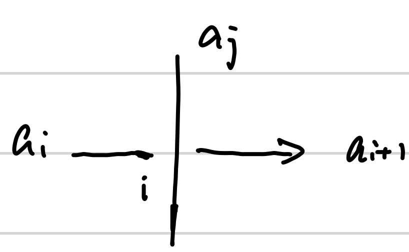
R2 

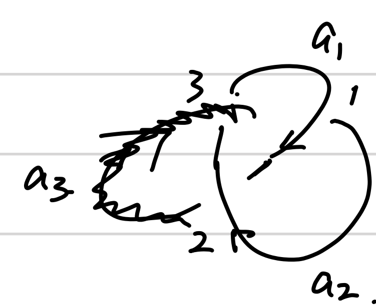
R3 

π_1 Wirtinger presentation.

$$\pi_1(S^3 \setminus K) \cong \langle a_1, \dots, a_n \mid w_1, \dots, w_n \rangle.$$

$$w_i = \begin{cases} a_i a_j^{-1} a_{i+1}^{-1} a_j, & + \text{ - crossing} \\ a_i a_j a_{i+1}^{-1} a_j^{-1}, & - \text{ - crossing.} \end{cases}$$



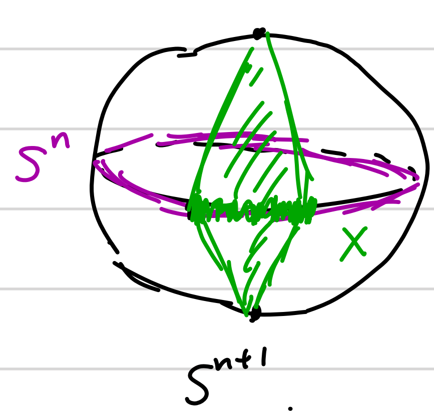


- Spanier - Whitehead duality:

  X determines the stable htpy type of $K := S^n \setminus X$.

  stable htpy type of X.

$X \subset S^n$. Embed S^n as an equatorial sphere in S^{n+1}.



ΣX = joining to the two poles.

$$S^{n+1} \setminus \Sigma X \simeq S^n \setminus X$$

$Y \subset S^m$ m not nec. equal to n.

$f' : \Sigma^p X \to \Sigma^q Y$. provided we have $f : X \to Y$

$X' \subset S^{n'} \hookrightarrow S^{n'+1}$ without changing X'

$S^{n'+1} \setminus \Sigma X' = S^{n'} \setminus X'$

Consider $S^{n'} * S^{m'} = S^{n'+m'+1}$

Recall X, Y

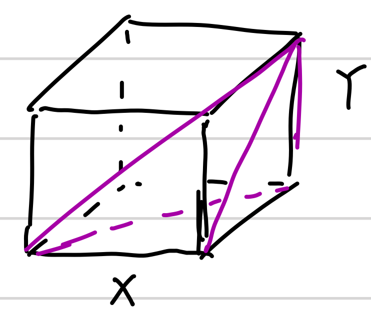
$X * Y$ = space of all line segments joining X to Y.

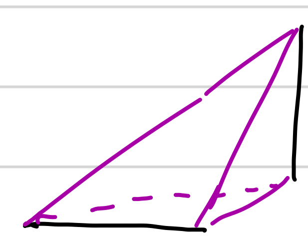
$= X \times Y \times I \Big/ \begin{array}{l} (x, y_1, 0) \sim (x, y_2, 0) \\ (x_1, y, 1) \sim (x_2, y, 1). \end{array}$

$X = $ ——— I

$Y = $ ——— I.

$X * Y$:





$X * Y$.

$f' : X \to Y$, $Y \subset S^{m'}$.

$X \subset S^{n'}$.

$S^{m'+n'+1} \setminus X \simeq \Sigma^{m'+1} (S^{n'} \setminus X)$

$S^{m'+n'+1} \setminus Y \simeq \Sigma^{n'+1} (S^{m'} \setminus Y)$

need verify !

$\Rightarrow M$ = mapping cylinder of f'.

$S^{n'+m'+1} \setminus X \xleftarrow{f} S^{m'+n'+1} \setminus M \xrightarrow{g} S^{n'+m'+1} \setminus Y$.

Injections $X \hookrightarrow M$ $\Rightarrow$ cohomology iso

$Y \nearrow$

$f, g \Rightarrow$ homology iso.

Alexander duality.

$\Rightarrow f, g$ stable htpy equiv.

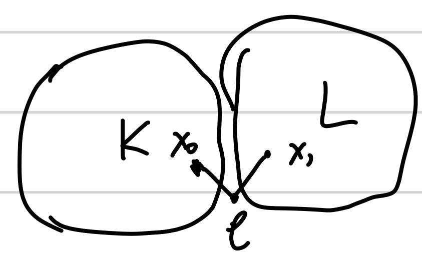
- $f: X \to Y$

$\updownarrow$

$f^*: S^? \setminus Y \to S^? \setminus X \rightsquigarrow$ stable class. contravariant fctor.

- $X \subset S^n$, replace X by finite subcomplexes K, L, $K \cap L = \phi$. $K \cup L = S^n$.

Choose PL path from pt $\in K$ to pt $\in L$.



first pt $\in K$. Last pt $\in L$.

x_1, x_0 is the only pt that meets L, K, respectively.

Choose $x_2 \in \ell$. to be ∞.

$\mu: K \times L \to S^{n-1}$

$(s, r) \longmapsto \frac{s-r}{\|s-r\|}$

$\mu|_{K \times \{x_1\}}$, $\mu|_{\{x_0\} \times L}$ null-htpic.

$\rightsquigarrow \mu: K \wedge L \to S^{n-1}$.

Work in Sp: X CW spectrum / finite spectrum.

- $[W \wedge X, \mathbb{S}]_0$

$\Rightarrow$ Brown functor, representable. by Brown's representability.

$\Rightarrow [W \wedge X, \mathbb{S}]_* \overset{\Phi}{\cong} [W, DX]_*$ nat. iso

If $W = DX$, $\Rightarrow \exists$ map. $\eta = \Phi(\mathrm{id}): DX \wedge X \to \mathbb{S}$.
$(\mathrm{id}: DX \to DX)$

$\forall f: W \to DX \Rightarrow \quad W \wedge X \xrightarrow{f \wedge \mathrm{id}} DX \wedge X \xrightarrow{\eta} \mathbb{S}$.

- $g: X \to Y$. $\exists$ nat. trans.

$$\begin{tikzcd}
{[W \wedge X, \mathbb{S}]} \arrow[d, "\cong"] & {[W \wedge Y, \mathbb{S}]} \arrow[l, "(\mathrm{id} \wedge g)^*"'] \arrow[d, "\cong"] \\
{[W, DX]} & {[W, DY]} \arrow[l, "g^*"']
\end{tikzcd}$$

induced by $g^*: DY \to DX$. s.t.

$$\begin{tikzcd} DY \wedge X \arrow[r, "id \wedge g"] \arrow[d, "g^* \wedge id"'] & DY \wedge Y \arrow[d, "\eta_Y"] \\ DX \wedge X \arrow[r, "\eta_X"'] & \mathbb{S} \end{tikzcd}$$

$\Rightarrow \forall\ Z \in Sp$, W, X finite.

adjoint : $[W, Z \wedge DX]_* \xrightarrow{\cong} [W \wedge X, Z]_*$.

- X finite spectrum, so is DX.
- $D^2X \simeq X$ b/c $X \wedge DX = DX \wedge X$

$$\begin{tikzcd} {[X \wedge DX, \mathbb{S}]} \arrow[d, "="] & {[DX, DX]} \arrow[l, "\cong"'] \\ {[DX \wedge X, \mathbb{S}]} \arrow[r, "\cong"] & {[X, D^2X]} \end{tikzcd}$$

- Pf. $DS^n = S^{-n}$.

  Since CW cpxes (finite) can be obtained by attaching cells / spheres
  $\Rightarrow$ dual of CW cpxes (finite) can also be built in the same manner.

  By $D\Sigma X = \Sigma^{-1} DX$ (Exercise !)

  $\Rightarrow$ $X \in Sp$ finite, $\exists\ DX$.

4. Bousfield Classes.

<u>Recall</u>. E_* generalized homology theory.

Y E_*-local if $\forall\ f: X_1 \to X_2$ s.t. E_*f iso, then

$$[X_1, Y] \xleftarrow{f^*} [X_2, Y] \text{ iso.}$$

X, E_*-localization of X, $L_E X$, is a map. $\eta: X \to L_E X$.

$L_E X$ E_*-local s.t. $E_*\eta$ iso.

<u>Prop</u>. 1. $W \to X \to Y \to \Sigma W$ cofib seq.

If W, X, Y 2-out-of-3 E_*-local.

2. $X \vee Y$ E_*-local, then X, Y ✓.

<u>Warning</u>. $\mathrm{colim}(X_1 \to X_2 \to \cdots)$ not. nec. E_*-local!

• <u>Thm</u>. (Bousfield localization). $\forall$ homology theory E_*. $\forall X$,
$\exists!$ (up to htpy) $L_E X$, $L_E(-)$ functorial.

<u>Lem</u>. E ring spectrum, then $E \wedge X$, E_*-local.

<u>Def</u>. E ring spectrum. $\mathcal{A}$ = class, satisfies:
1) $E \in \mathcal{A}$.
2) If $N \in \mathcal{A}$, $N \wedge X \in \mathcal{A}$. $\forall X$.
3) $\forall f: X \to Y$, $X, Y \in \mathcal{A}$, $\mathrm{cofib}\, f \in \mathcal{A}$.
4) $X \in \mathcal{A}$. retract of $X \in \mathcal{A}$.

$\forall X \in \mathcal{A}$, called E_*-nilpotent. $X \in Sp$. X E_*-equiv to $N \in \mathcal{A}$.
Call X E_*-prenilpotent.

<u>Cor</u>. X E_*-nilpotent $\Rightarrow$ X E_*-local.

<u>Def</u>. $E, F \in Sp$. E, F called Bousfield equiv if $\forall X$,
$E \wedge X$ contractible $\Leftrightarrow$ $F \wedge X$ contractible.
Let $\langle E \rangle$ = Bousfield class of E
= equiv. class of Bousfield equiv of E.
▶ $\langle E \rangle = \{ X \in Sp : E \wedge X = 0 \}$.

<u>Nota</u>. $\langle E \rangle \geq \langle F \rangle$ if $\forall X$, $E \wedge X = 0 \Rightarrow F \wedge X = 0$.
$\langle E \rangle > \langle F \rangle$ if $\langle E \rangle \neq \langle F \rangle$ & $\langle E \rangle \geq \langle F \rangle$.

<u>Prop</u>. $\langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$
$\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle$

Prop. $(\langle X\rangle \vee \langle Y\rangle) \wedge \langle Z\rangle = (\langle X\rangle \wedge \langle Z\rangle) \vee (\langle Y\rangle \wedge \langle Z\rangle)$
$(\langle X\rangle \wedge \langle Y\rangle) \vee \langle Z\rangle = (\langle X\rangle \vee \langle Z\rangle) \wedge (\langle Y\rangle \vee \langle Z\rangle)$.

Prop. $L_E = L_F$ if $\langle E\rangle = \langle F\rangle$.
If $\langle E\rangle \le \langle F\rangle$, then $L_E L_F = L_E$. $\exists$ nat. trans. $L_F \to L_E$.

Cor. $\forall\ E \in Sp$.
1) $\langle S\rangle \ge \langle E\rangle \ge \langle pt.\rangle$.
2) $\langle S\rangle \wedge \langle E\rangle = \langle E\rangle$.
3) $\langle S\rangle \vee \langle E\rangle = \langle S\rangle$
4) $\langle pt\rangle \wedge \langle E\rangle = \langle pt\rangle$.
5) $\langle pt\rangle \vee \langle E\rangle = \langle E\rangle$.

- $\langle S\rangle$ biggest, $\langle pt\rangle$ smallest
"$\ge$"

__Ohkawa's theorem.__ $\{\langle E\rangle : E \in Sp\}$ is a set.
哲介

Prop 1) $f: X \to Y$. $W \to X \xrightarrow{f} Y \to \Sigma W$ cofib seq. Then
$\langle W\rangle \le \langle X\rangle \vee \langle Y\rangle$.
If f smash nilpotent, "$\le$" becomes "$=$".
2). $f: X \to \Sigma^{-d}X$ self map, $C_f = \text{cofib } f$.
$\hat{X} = \operatorname{colim}_i \Sigma^{-id}X$
$= \operatorname{colim}\,(X \xrightarrow{f} \Sigma^{-d}X \xrightarrow{f} \Sigma^{-2d}X \xrightarrow{f} \cdots)$.
Then $\langle X\rangle = \langle \hat{X}\rangle \vee \langle C_f\rangle$
$\langle \hat{X}\rangle \wedge \langle C_f\rangle = \langle pt.\rangle$

Thm. (Class invariance).
X, Y p-local finite of type m, n, respectively. Then
$\langle X\rangle = \langle Y\rangle \iff m = n$.

$$\langle X \rangle < \langle Y \rangle \iff m > n.$$

pf. Assume we have proven TST!

Let $\mathcal{C}_X$, $\mathcal{C}_Y$ be smallest thick subcats of $FH_{(p)}$ containing X, Y, respectively. $\forall X' \in \mathcal{C}_X$, $K(m-1)_*(X') = 0$.

TST $\Rightarrow$

$$FH_{(p)} = \mathcal{D}_0 \supset \mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots$$
$$\mathcal{C}_X = \mathcal{D}_m$$

Similarly, $\mathcal{C}_Y = \mathcal{D}_n$.

If $m = n$, $\mathcal{D}_m = \mathcal{D}_n \iff \mathcal{C}_X = \mathcal{C}_Y \iff \langle X \rangle = \langle Y \rangle$.

If $m > n$, $\mathcal{D}_m \subset \mathcal{D}_n \iff \langle X \rangle < \langle Y \rangle$.

Prop. $S_{\mathbb{Q}}$ rational sphere spectrum

$S_{(p)}$ p-local ——

S/p mod p Moore spectrum.

Then
1). $\langle S_{(p)} \rangle = \langle S_{\mathbb{Q}} \rangle \vee \langle S/p \rangle$
2) $\langle S_{\mathbb{Q}} \rangle \wedge \langle S/p \rangle = \langle pt. \rangle$.
3) $\langle S/p \rangle \wedge \langle S/q \rangle = \langle pt. \rangle$. $p \neq q$.
4) $\langle S \rangle = \langle S_{\mathbb{Q}} \rangle \vee \bigvee_{p \text{ prime}} \langle S/p \rangle$.

$MU_{(p)} = \bigvee \Sigma^{?} BP$ leads to.

- $\langle MU_{(p)} \rangle = \langle BP \rangle$ [Rav 84. Section 2]
- $\langle MU \rangle = \bigvee_{p \text{ prime}} \langle MU_{(p)} \rangle = \bigvee_{p \text{ prime}} \langle BP \rangle$.

• Th. [Rav 84. Section 2-4].

1. (Johnson-Wilson) $\langle B(n) \rangle = \langle K(n) \rangle$.

$P(n) = BP / I_n$, $I_n = (p, v_1, \cdots, v_{n-1})$.

$P(0) = BP$. $P(n)_* = BP_* / I_n$.

$\Sigma^{2(p^n-1)} P(n) \xrightarrow{v_n} P(n) \longrightarrow P(n+1)$.

$B(n) = \operatorname{colim}\left(P(n) \xrightarrow{v_n} \Sigma^{-2(p^n-1)} P(n) \xrightarrow{v_n} \Sigma^{-4(p^n-1)} P(n) \xrightarrow{v_n} \cdots\right)$

$BP\langle n\rangle = BP/(v_{n+1}, v_{n+2}, \dots)$.

$E(n) = \mathrm{colim}\,(BP\langle n\rangle \xrightarrow{v_n} \Sigma^{-2(p^n-1)} BP\langle n\rangle \xrightarrow{v_n} \cdots)$.

2. (Johnson – Yosimura) $\langle v_n^{-1}BP\rangle = \langle E(n)\rangle$.
3. $\langle P(n)\rangle = \langle K(n)\rangle \vee \langle P(n+1)\rangle$.
4. $\langle E(n)\rangle = \bigvee_{i=0}^{n} \langle K(i)\rangle$.
5. $\langle BP\langle n\rangle\rangle = \langle E(n)\rangle \vee \langle H\mathbb{Z}/p\rangle$.
6. $\langle K(m)\rangle \wedge \langle K(n)\rangle = \langle pt.\rangle$ if $m \neq n$.

- $v_n^{-1}BP_*(X) = 0 \iff K(n)_*(X) = 0$:

  $\langle v_n^{-1}BP\rangle = \langle E(n)\rangle = \bigvee_{i=0}^{n} \langle K(i)\rangle$.

5. Proof of Key Corollary

★ <u>Key Corollary</u>

W, X, Y p-local finite spectra, $f: X \to Y$.

If $K(n)_*(W\wedge f) = 0$, $n \geq 0$, then $W \wedge f^{(k)}$ null-htpic, $k >> 0$.

<u>Lem ★</u>. $\forall\, k > 1$, $\exists$ cofib seq.

$$C_{f^{(k)}} \to C_{f^{(k-1)}} \to \Sigma W^{(k-1)} \wedge C_f ,$$

Here we assume X finite, $f: W \to S$ s.t. $W \xrightarrow{f} S \xrightarrow{c} DX \wedge X$

cofib seq. $C_f = C_{f^{(1)}} = DX \wedge X$.

<u>Pf. of Lem ★</u>. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$.

$\exists$ comm. diag.

$$\begin{tikzcd}
C_f \arrow[r] & pt. \arrow[r] & \Sigma C_f \\
Y \arrow[u] \arrow[r, "g"] & Z \arrow[u] \arrow[r] & C_g \arrow[u] \\
X \arrow[u, "f"] \arrow[r, "gf"] & Z \arrow[u, "id"] \arrow[r] & C_{gf} \arrow[u]
\end{tikzcd}$$

s.t. each col & row is a cofib seq.

Let $X = W^{(k)}$ $\qquad g = f^{(k-1)}$
$Y = W^{(k-1)}$
$Z = \mathbb{S}$

$$\begin{tikzcd}
W^{(k-1)} \wedge C_f & W^{(k-1)} \wedge C_f \arrow[r] & \mathrm{pt.} \arrow[r] & \Sigma W^{(k-1)} \wedge C_f \\
W^{(k-1)} \wedge \mathbb{S} \arrow[u] \arrow[r, equal] & W^{(k-1)} \arrow[u] \arrow[r, "f^{(k)}"] & \mathbb{S} \arrow[u] \arrow[r] & C_{f^{(k-1)}} \arrow[u] \\
W^{(k-1)} \wedge W \arrow[u, "f"] \arrow[r, equal] & W^{(k)} \arrow[u, "f"] \arrow[r, "f^{(k)}"] & \mathbb{S} \arrow[u, "\mathrm{id}"] \arrow[r] & C_{f^{(k)}} \arrow[u]
\end{tikzcd}$$

this is what we're looking for!

$\square$

pf of Key Corollary.

Write $R = DW \wedge W$.

$e : \mathbb{S} \to R \quad \dashv \quad \eta : R \to \mathbb{S}$ identity

$\Rightarrow \quad D\eta : R \to \mathbb{S}$

R ring spectrum, unit e.

multiplication $m : R \wedge R \to DW \wedge W \wedge DW \wedge W \xrightarrow{\mathrm{id} \wedge \eta \wedge \mathrm{id}} DW \wedge \mathbb{S} \wedge W \to DW \wedge W = R.$

We know: $f : X \to Y$
$\wr$ adjoint
$\tilde{f} : \mathbb{S} \to DX \wedge Y$

$W \wedge f$ adjoint to $\left(\mathbb{S} \xrightarrow{\tilde{f}} DX \wedge Y \xrightarrow{e \wedge \mathrm{id} \wedge \mathrm{id}} R \wedge DX \wedge Y =: F \right)$

$DX \wedge Y = \mathbb{S} \wedge DX \wedge Y$

!!

g.

Then $W \wedge f^{(k)}$ adjoint to $g^{(k)}$, h/c.

$W \wedge f^{(k)} : W \wedge X^{(k)} \longrightarrow W \wedge Y^{(k)}$

$$S \xrightarrow{g^{(k)}} F^{(k)} = \underbrace{R^{(k)}} \wedge DX^{(k)} \wedge Y^{(k)}$$
$$\xrightarrow{m^{k-1}} R \wedge DX^{(k)} \wedge Y^{(k)}.$$

To show $W \wedge f^{(k)}$ is null-htpic $\Longleftrightarrow$ $g^{(k)}$ null-htpic.

By ~~nilpotence theorem~~, smash product form.

$F \xrightarrow{f} X$, F finite, f smash nilpotent if $MU \wedge f$ null-htpic
(BP)

It suffices to prove $BP \wedge g^{(k)}$ is null-htpic for $k >> 0$.

Let $T_k = R \wedge DX^{(k)} \wedge Y^{(k)}$

$T = \mathrm{colim}\,(S \xrightarrow{g} T_1 \xrightarrow{id \wedge \tilde{f}} T_2 \xrightarrow{id \wedge \tilde{f}} \cdots).$

Suffice to prove $BP \wedge T$ contractible.

$$\langle BP \rangle = \langle K(0) \rangle \vee \langle K(1) \rangle \vee \cdots \vee \langle K(n) \rangle \vee \langle P(n+1) \rangle$$

[Rav. 84, Section 2].

$K(n) \wedge T$ contractible. $\forall n$.

Suffice to prove $P(m) \wedge T$ contractible. $m >> 0$.

$$K(m)_*(W \wedge f) = K(m)_* \otimes_{K(m)_*} H_*(W \wedge f).$$
$$P(m)_*(W \wedge f) = P(m)_* \otimes_{P(m)_*} H_*(W \wedge f).$$

both $0 \Rightarrow \checkmark$.

6. Proof of TST.

$\mathcal{C} \subset FH_{(p)}$ thick subcat.

n = smallest integer s.t. $\mathcal{C}$ contains all p-local finite spectrum X s.t.
$K(n)_*(X) \neq 0 \Longleftrightarrow v_n^{-1} BP_*(X) = 0$. Goal. $\mathcal{C} = \mathcal{D}_{p,n}$.
$= \mathcal{D}_n$ (p fixed).

$\mathcal{C} \subset \mathcal{D}_n$. Suffice to prove "$\supset$".

Let $Y \in \mathcal{D}_n$. p-local finite spectrum. $\mathcal{C}$ thick $\Rightarrow$ $X \wedge F \in \mathcal{C}$.
F finite

$\Rightarrow$ $X \wedge DX \wedge Y \in \mathcal{C}$.

Lem ★. $f : W \to \mathbb{S}$
W finite $\in FH_{(p)}$.
$W \xrightarrow{f} \mathbb{S} \longrightarrow C_f = X \wedge DX$ cofib seq.
$\Rightarrow$ $C_{f^{(k)}} \wedge Y \in \mathcal{C}$. $\forall k > 0$

Recall $\mathrm{Hom}(K(n)_*(X), K(n)_*(Y)) = K(n)_*(DX \wedge Y)$.
$\Rightarrow$ $K(i)_*(f) = 0$ b/c $DW \in FH_{(p)} = \mathcal{D}_0$.
$K(i)_*(f) = K(i)_*(DW)$ $i \geq n$.
Since $K(i)_*(Y) = 0$, $i < n$.
$\Rightarrow$ $K(i)_*(Y \wedge f) = 0$, $\forall i$
By Key Corollary $\Rightarrow$ $Y \wedge f^{(k)}$ null-htpic. $k \gg 0$.

- cofib of null htpic map equiv to the wedge of its target & Σ of its source. (Exercise).

$Y \wedge C_{f^{(k)}} \simeq Y \vee (\Sigma Y \wedge W^{(k)})$.

$\Rightarrow$ $Y \in \mathcal{D}_n$ and $Y \in \mathcal{C}$.
$\Rightarrow$ $\mathcal{D}_n \subset \mathcal{C}$.
$\Rightarrow$ $\mathcal{C} = \mathcal{D}_n$.

$\square$